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Circularly cylindrical layered media in plane elasticity

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Abstract

This paper presents an alternative efficient procedure to analyze plane elasticity problems of a circularly cylindrical layered media subject to an arbitrary point force. Based on the method of analytical continuation in conjunction with the alternating technique, the elastic fields of the three-phase media are derived. A rapidly convergent series solution which is expressed in terms of an explicit general term of the complex potential of the corresponding homogeneous problem is obtained in an elegant form. As a numerical illustration, the interfacial stresses are presented for different material combinations and for different positions of the point force.

Keywords: Cylindrical layered media; Plane elasticity; Continuation theorem; Alternating technique

1. Introduction

In view of the rapidly increasing use of composite materials in many engineering applications, considerable research activities in the area of stress analysis of layered medium have been of significant concern in recent years. The interaction between singularities and multiple-phase materials becomes an important topic in studying the damage mechanism of composite structures. Because of the inherent heterogeneous nature of the composite, the analysis of such materials is much more involved than that of homogeneous counterparts. For multi-layered composites, the problem becomes more complicated since solutions to the elasticity problem for all layers are required. Consequently, the conventional procedure of stress analysis of multi-layered media results in having to solve a system of simultaneous equations for a large number of

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unknown constants. The complexity of such a procedure can be found in the work of Iyengar and Alwar (1964) as well as Chen (1971) who analyzed the semi-infinite medium composed of isotropic layers. As an alternative efficient approach to the analysis of multi-layered media, various solution procedures have been developed. Bufler (1971) used the transfer matrix approach to convert the boundary value problem to an equivalent initial value problem based on the mixed formulation of elasticity proposed by Vlasov and Leontev (1966). This transfer matrix is expressed in terms of the infinite series expansion allowing solutions with various orders of approximation to be obtained. Based on the flexibility matrix method, Small and Booker (1984) performed the stress analysis of a layered medium resting on a rigid foundation. This method has been found to have an advantage of significant reducing the number of simultaneous equations. Lin and Keer (1989) also used the flexibility matrix method together with the boundary integral formulation to deal with a vertical crack in a layer medium. Based on the Fourier transform technique in conjunction with the stiffness matrix approach, Choi and Thangjitham (1991) obtained the solutions of multi-layered anisotropic elastic media. Choi and Earmme (2002a,b) employed the alternating technique to obtain the solution of singularity problems in an isotropic and anisotropic plane layered trimaterial. However, for the analogous problems of multi-layered media with circular interfaces, more mathematical difficulties are encountered. Based on the Laurent series expansion, Luo (1991) found a solution for an edge dislocation in a three-phase composite cylinder. Their results are valid only for the case that an edge dislocation (or singularity) is situated at the intermediate annular region of composite structure.

In this paper we consider the problem of an isotropic three-phase circularly cylindrical media interacted with an arbitrary point force. A point force (or singularity) considered in this paper is located either in the matrix or in the inclusion. The proposed method is based on the technique of analytical continuation that is alternatively applied across the two concentric circular interfaces in order to derive the trimaterial solution in a series form from the corresponding homogeneous solution. The plane of the paper is as follows. The general formulation for plane isotropic elasticity is provided in Section 2. The general form of the complex potentials of the stress functions for a trimaterial is provided in Section 3. Some numerical results are discussed in Section 4. Finally, Section 5 concludes the article.

2. Isotropic elasticity

For a two-dimensional theory of elasticity, the components of displacement and traction force can be expressed in terms of two stress functions $\phi(z)$ and $\psi(z)$ as (Muskhelishvili, 1953)

$$2G(u+\mathrm{i}v) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{1}$$

$$-Y+\mathrm{i}X = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{2}$$

where G is the shear modulus, $\kappa = 3-4\nu$, for plane strain and $(3-\nu)/(1+\nu)$, for plane stress with ν being the Poisson's ratio. Here a superimposed bar represents the complex conjugate. The components stress in polar coordinates are related to $\phi(z)$ and $\psi(z)$ by (Muskhelishvili, 1953)

$$\sigma_{rr} + \sigma_{\theta\theta} = 2[\phi'(z) + \overline{\phi'(z)}] \tag{3}$$

$$\sigma_{rr} + \mathrm{i}\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} - \left[\bar{z}\overline{\phi''(z)} + \frac{\bar{z}}{z}\overline{\psi'(z)}\right] \tag{4}$$

For the problem associated with an isotropic elastic bimaterial, the stresses are found to depend on only two non-dimensional Dundurs parameters (Dundurs, 1969)

$$\alpha_{ab} = \frac{G_a(\kappa_b+1) - G_b(\kappa_a+1)}{G_a(\kappa_b+1) + G_b(\kappa_a+1)}, \quad \beta_{ab} = \frac{G_a(\kappa_b-1) - G_b(\kappa_a-1)}{G_a(\kappa_b+1) + G_b(\kappa_a+1)} \tag{5}$$

where a and b refer to the two materials composing the bimaterial. Another pairs associated with the above two parameters are defined as

$$\Lambda_{ab} = \frac{\alpha_{ab} + \beta_{ab}}{1 - \beta_{ab}}, \quad \Pi_{ab} = \frac{\alpha_{ab} - \beta_{ab}}{1 + \beta_{ab}} \tag{6}$$

which will be used in our subsequent derivations for trimaterial problems.

Consider a point force of magnitude F enclosing an angle γ with the x_1-axis embedded in a point $z_0 = r_0 e^{i\theta_0}$ of an infinite homogeneous medium, the solutions are (Muskhelishvili, 1953)

$$\phi_0(z) = -\frac{F e^{i\gamma}}{2\pi(1+\kappa)} \log(z - z_0) \tag{7}$$

$$\psi_0(z) = \frac{\kappa F e^{-i\gamma}}{2\pi(1+\kappa)} \log(z - z_0) + \frac{F e^{i\gamma}}{2\pi(1+\kappa)} \frac{\overline{z_0}}{z - z_0} \tag{8}$$

These fields will be used for the corresponding problem of the same singularity in a trimaterial in the following sections.

3. A singularity in a trimaterial and the alternating technique

To analyze a singularity in a trimaterial with two concentric circular interfaces as shown in Fig. 1, the alternating technique together with the method of analytical continuation is applied. Since it is difficult to find a solution satisfying all the continuity conditions along two interfaces at the same time, the method of analytical continuation should be applied to two interfaces alternatively.

For a region bounded by a circle, say $c = |z|$, Eqs. (1), (2) and (4), respectively can be rewritten as

$$2G(u + iv) = \kappa\phi(z) - \overline{\omega(z)} + \left(\frac{c^2}{\bar{z}} - z\right)\overline{\phi'(z)} \tag{9}$$

$$-Y + iX = \phi(z) + \overline{\omega(z)} + \left(z - \frac{c^2}{\bar{z}}\right)\overline{\phi'(z)} \tag{10}$$

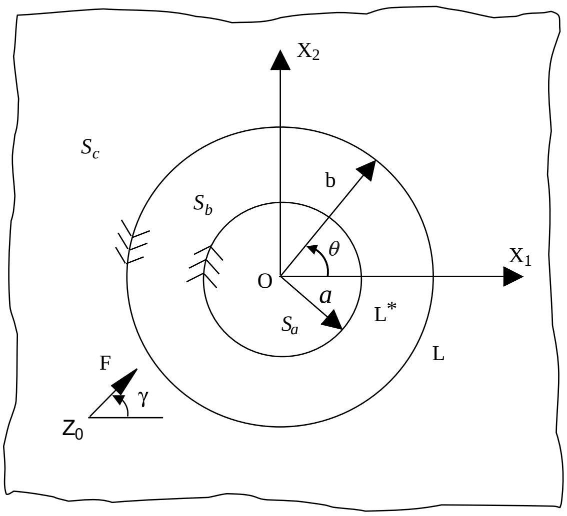

Fig. 1. A point force (or singularity) in a trimaterial.

$$\sigma_{rr} + \mathrm{i}\sigma_{r\theta} = \phi'(z) - \frac{\bar{z}}{z}\overline{\omega'(z)} + \left(1 - \frac{c^2}{z\bar{z}}\right)\overline{\phi'(z)} + \left(\frac{c^2}{z} - \bar{z}\right)\overline{\phi''(z)} \tag{11}$$

where

$$\omega(z) = \frac{c^2}{z}\phi'(z) + \psi(z) \tag{12}$$

3.1. Case I: A point force embedded in S_c

Assume that regions S_a, S_b and S_c occupied with material a, b and c, respectively are perfectly bonded along the interfaces $r = a$ and $r = b$ (see Fig. 1). The alternating technique is applied to solve the problem of a trimaterial subjected a singularity in region S_c by considering the following steps.

First, we regard regions S_a and S_b composed of the same material b and region S_c of material c. $\phi_1(z)$ and $\omega_1(z)$ holomorphic (except at $z = 0$) in $S_a \cup S_b$, $\phi_{c0}(z)$ and $\omega_{c0}(z)$ holomorphic in S_c are introduced to satisfy the continuity of traction and displacement across L that

$$\phi_1(\rho) + \overline{\omega_1(\rho)} = \phi_0(\rho) + \overline{\omega_0(\rho)} + \phi_{c0}(\rho) + \overline{\omega_{c0}(\rho)} \tag{13}$$

$$\frac{1}{G_b}\left[\kappa_b\phi_1(\rho) - \overline{\omega_1(\rho)}\right] = \frac{1}{G_c}\left[\kappa_c\phi_{c0}(\rho) + \kappa_c\phi_0(\rho) - \overline{\omega_{c0}(\rho)} - \overline{\omega_0(\rho)}\right] \tag{14}$$

where $\rho = b\mathrm{e}^{\mathrm{i}\theta}$ and

$$\omega_0(z) = \frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_c)}\left(\overline{z_0} - \frac{b^2}{z}\right)\frac{1}{(z - z_0)} + \frac{\kappa_c F\mathrm{e}^{-\mathrm{i}\gamma}}{2\pi(1+\kappa_c)}\log(z - z_0) \tag{15}$$

By the standard analytical continuation arguments it follows that

$$\overline{\omega_1}\left(\frac{b^2}{z}\right) - \phi_{c0}(z) - \overline{\omega_0}\left(\frac{b^2}{z}\right) - C_1 z + C_0 z = 0, \quad z \in S_c \tag{16}$$

$$\phi_0(z) + \overline{\omega_{c0}}\left(\frac{b^2}{z}\right) - \phi_1(z) - C_1 z + C_0 z = 0, \quad z \in S_a \cup S_b \tag{17}$$

$$-\frac{\overline{\omega_1}\left(\frac{b^2}{z}\right)}{G_b} - \frac{\kappa_c}{G_c}\phi_{c0}(z) + \frac{\overline{\omega_0}\left(\frac{b^2}{z}\right)}{G_c} + \frac{C_1 z}{G_b} - \frac{C_0 z}{G_c} = 0, \quad z \in S_c \tag{18}$$

$$\frac{\kappa_c}{G_c}\phi_0(z) - \frac{\overline{\omega_{c0}}\left(\frac{b^2}{z}\right)}{G_c} - \frac{\kappa_b}{G_b}\phi_1(z) + \frac{C_1 z}{G_b} - \frac{C_0 z}{G_c} = 0, \quad z \in S_a \cup S_b \tag{19}$$

Solve Eqs. (16)–(19) to obtain

$$\phi_1(z) = (1 + \Lambda_{bc})\phi_0(z) + \Pi_{cb} C_1 z, \quad z \in S_a \cup S_b \tag{20}$$

$$\omega_1(z) = (1 + \Pi_{bc})\omega_0(z) - (1 + \Pi_{bc})\overline{C_0}\frac{b^2}{z} + \overline{C_1}\frac{b^2}{z}, \quad z \in S_a \cup S_b \tag{21}$$

$$\phi_{c0}(z) = \Pi_{bc}\overline{\omega_0}\left(\frac{b^2}{z}\right) - \Pi_{bc} C_0 z, \quad z \in S_c \tag{22}$$

$$\omega_{c0}(z) = \Lambda_{bc}\overline{\phi_0}\left(\frac{b^2}{z}\right) + (1 + \Pi_{cb})\overline{C_1}\frac{b^2}{z} - \overline{C_0}\frac{b^2}{z}, \quad z \in S_c \tag{23}$$

where $C_0 = \overline{\phi_0'}(0)$ and $C_1 = \overline{\phi_1'}(0)$.

Since this result is based on the assumption that region S_a is made up of material b, it cannot satisfy the continuity condition across L^*, which lies between material a and b.

In the second step, we assume region S_b and S_c be made up of the same material b and region S_a of material a. Additional terms $\phi_{b1}(z)$ and $\omega_{b1}(z)$ holomorphic in $S_b \cup S_c$, $\phi_{a1}(z)$ and $\omega_{a1}(z)$ holomorphic (except at $z = 0$) in S_a are introduced to satisfy the continuity conditions across L^* that

$$\phi_{a1}(\sigma) + \overline{\omega_{a1}(\sigma)} = \phi_1(\sigma) + \phi_{b1}(\sigma) + \overline{\omega_1^a(\sigma)} + \overline{\omega_{b1}(\sigma)} \tag{24}$$

$$\frac{\kappa_a\phi_{a1}(\sigma) - \overline{\omega_{a1}(\sigma)}}{G_a} = \frac{\kappa_b[\phi_{b1}(\sigma) + \phi_1(\sigma)] - [\overline{\omega_{b1}(\sigma)} + \overline{\omega_1^a(\sigma)}]}{G_b} \tag{25}$$

where $\omega_1^a(z) = \omega_1(z) + \frac{(a^2-b^2)}{z}\phi_1'(z)$ and $\sigma = a\mathrm{e}^{\mathrm{i}\theta}$.

Based on the method of analytical continuation, we have

$$\phi_{a1}(z) - \phi_1(z) - \overline{\omega_{b1}}\left(\frac{a^2}{z}\right) - C_1 z + C_{a1} z = 0, \quad z \in S_a \tag{26}$$

$$\phi_{b1}(z) + \overline{\omega_1^a}\left(\frac{a^2}{z}\right) - \overline{\omega_{a1}}\left(\frac{a^2}{z}\right) - C_1 z + C_{a1} z = 0, \quad z \in S_b \cup S_c \tag{27}$$

$$\frac{\kappa_a\phi_{a1}(z)}{G_a} - \frac{\kappa_b\phi_1(z) - \overline{\omega_{b1}}\left(\frac{a^2}{z}\right)}{G_b} - \frac{C_{a1}z}{G_a} + \frac{C_1 z}{G_b} = 0, \quad z \in S_a \tag{28}$$

$$\frac{\kappa_b\phi_{b1}(z) - \overline{\omega_1^a}\left(\frac{a^2}{z}\right)}{G_b} + \frac{\overline{\omega_{a1}}\left(\frac{a^2}{z}\right)}{G_a} - \frac{C_{a1}z}{G_a} + \frac{C_1 z}{G_b} = 0, \quad z \in S_b \cup S_c \tag{29}$$

Solve Eqs. (26)–(29) to obtain

$$\phi_{a1}(z) = (1 + \Lambda_{ab})\phi_1(z) + \Pi_{ba} C_{a1} z, \quad z \in S_a \tag{30}$$

$$\omega_{a1}(z) = (1 + \Pi_{ab})\omega_1^a(z) - (1 + \Pi_{ab})\overline{C_1}\frac{a^2}{z} + \overline{C_{a1}}\frac{a^2}{z}, \quad z \in S_a \tag{31}$$

$$\phi_{b1}(z) = \Pi_{ab}\overline{\omega_1^a}\left(\frac{a^2}{z}\right) - \Pi_{ab} C_1 z, \quad z \in S_b \cup S_c \tag{32}$$

$$\omega_{b1}(z) = \Lambda_{ab}\overline{\phi_1}\left(\frac{a^2}{z}\right) + (1 + \Pi_{ba})\overline{C_{a1}}\frac{a^2}{z} - \overline{C_1}\frac{a^2}{z}, \quad z \in S_b \cup S_c \tag{33}$$

where $C_{a1} = \overline{\phi_{a1}'}(0)$.

Since this result is based on the assumption that region S_c is made up of material b, it cannot satisfy the continuity conditions across L.

In the third step, we again regard regions S_a and S_b composed of the same material b and region S_c of material c. Additional terms $\phi_2(z)$, $\omega_2(z)$ holomorphic (except at $z = 0$) in $S_a \cup S_b$ and $\phi_{c1}(z)$, $\omega_{c1}(z)$ holomorphic in S_c are introduced to satisfy the continuity conditions across L as

$$\phi_{b1}(\rho) + \phi_2(\rho) + \overline{\omega_{b1}^b(\rho)} + \overline{\omega_2(\rho)} = \phi_{c1}(\rho) + \overline{\omega_{c1}(\rho)} \tag{34}$$

$$\frac{1}{G_b}\{\kappa_b[\phi_{b1}(\rho) + \phi_2(\rho)] - [\overline{\omega_{b1}^b(\rho)} + \overline{\omega_2(\rho)}]\} = \frac{1}{G_c}[\kappa_c\phi_{c1}(\rho) - \overline{\omega_{c1}(\rho)}] \tag{35}$$

where $\omega_{b1}^b(z) = \omega_{b1}(z) + \frac{(b^2-a^2)}{z}\phi_{b1}'(z)$.

By applying the method of analytical continuation, we have

$$\phi_{b1}(z) + \overline{\omega_2}\left(\frac{b^2}{z}\right) - \phi_{c1}(z) - C_2 z = 0, \quad z \in S_c \tag{36}$$

$$\overline{\omega_{c1}}\left(\frac{b^2}{z}\right)-\phi_2(z)-\overline{\omega_{b1}^b}\left(\frac{b^2}{z}\right)-C_2z=0,\quad S_a\cup S_b \tag{37}$$

$$\frac{1}{G_b}\left[\kappa_b\phi_{b1}(z)-\overline{\omega_2}\left(\frac{b^2}{z}\right)\right]-\frac{\kappa_c}{G_c}\phi_{c1}(z)+\frac{C_2z}{G_b}=0,\quad z\in S_c \tag{38}$$

$$-\frac{1}{G_c}\overline{\omega_{c1}}\left(\frac{b^2}{z}\right)-\frac{1}{G_b}\left[\kappa_b\phi_2(z)-\overline{\omega_{b1}^b}\left(\frac{b^2}{z}\right)\right]+\frac{C_2z}{G_b}=0,\quad S_a\cup S_b \tag{39}$$

Solve Eqs. (36)–(39) to obtain

$$\phi_2(z)=\Pi_{cb}\overline{\omega_{b1}^b}\left(\frac{b^2}{z}\right)+\Pi_{cb}C_2z,\quad z\in S_a\cup S_b \tag{40}$$

$$\omega_2(z)=\Lambda_{cb}\overline{\phi_{b1}}\left(\frac{b^2}{z}\right)+\overline{C_2}\frac{b^2}{z},\quad z\in S_a\cup S_b \tag{41}$$

$$\phi_{c1}(z)=(1+\Lambda_{cb})\phi_{b1}(z),\quad z\in S_c \tag{42}$$

$$\omega_{c1}(z)=(1+\Pi_{cb})\omega_{b1}^b(z)+(1+\Pi_{cb})\overline{C_2}\frac{b^2}{z},\quad z\in S_c \tag{43}$$

where $C_2=\overline{\phi_2'}(0)$.

In the fourth step, regions S_b and S_c are assumed to make up with material b again. Repetitions of second and third step, the analytical continuation method is alternatively applied to two interfaces to obtain the additional terms $\phi_{an}(z)$, $\phi_{bn}(z)$, $\phi_{cn}(z)$, $\phi_{n+1}(z)$ and $\omega_{an}(z)$, $\omega_{bn}(z)$, $\omega_{cn}(z)$, $\omega_{n+1}(z)$ ($n=2,3,4,\ldots$). The stress functions can be finally obtained as

$$\phi(z)=\begin{cases}\sum_{n=1}^{\infty}\phi_{an}(z), & z\in S_a\\ \sum_{n=1}^{\infty}[\phi_n(z)+\phi_{bn}(z)], & z\in S_b\\ \phi_0(z)+\phi_{c0}(z)+\sum_{n=1}^{\infty}\phi_{cn}(z), & z\in S_c\end{cases} \tag{44a}$$

$$\omega(z)=\begin{cases}\sum_{n=1}^{\infty}\omega_{an}(z), & z\in S_a\\ \sum_{n=1}^{\infty}[\omega_n(z)+\omega_{bn}(z)], & z\in S_b\\ \omega_0(z)+\omega_{c0}(z)+\sum_{n=1}^{\infty}\omega_{cn}(z), & z\in S_c\end{cases} \tag{44b}$$

If one expresses the stress functions $\phi(z)$ and $\omega(z)$ in terms of $\phi_0(z)$ and $\omega_0(z)$ respectively, Eq. (44) becomes

$$\phi(z)=\begin{cases}(1+\Lambda_{ab})\sum_{n=1}^{\infty}\phi_n(z)+\frac{\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi_{ba}^2}\sum_{n=1}^{\infty}C_nz+\frac{\Pi_{ba}^2(1+\Lambda_{ab})}{1-\Pi_{ba}^2}\sum_{n=1}^{\infty}\overline{C_n}z, & z\in S_a\\ \sum_{n=1}^{\infty}\phi_n(z)+\Lambda_{cb}^{-1}\left[\sum_{n=1}^{\infty}\overline{\omega_{n+1}\left(\frac{b^2}{z}\right)}-\sum_{n=1}^{\infty}C_{n+1}z\right], & z\in S_b\\ \phi_0(z)+\Pi_{bc}\overline{\omega_0}\left(\frac{b^2}{z}\right)-\Pi_{bc}C_0z+(1+\Lambda_{cb}^{-1})\left[\sum_{n=1}^{\infty}\overline{\omega_{n+1}\left(\frac{b^2}{z}\right)}-\sum_{n=1}^{\infty}C_{n+1}z\right], & z\in S_c\end{cases} \tag{45a}$$

$$\omega(z)=\begin{cases}(1+\Pi_{ab})\left[\sum_{n=1}^{\infty}\omega_n(z)+\frac{a^2-b^2}{z}\sum_{n=1}^{\infty}\phi_n'(z)\right]\\ \quad+\frac{\Lambda_{ab}-\Pi_{ab}+\Pi_{ba}^2(1+\Pi_{ab})}{1-\Pi_{ba}^2}\sum_{n=1}^{\infty}\overline{C_n}\frac{a^2}{z}+\frac{\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi_{ba}^2}\sum_{n=1}^{\infty}C_n\frac{a^2}{z}, \quad z\in S_a\\ \sum_{n=1}^{\infty}\omega_n(z)+\Pi_{cb}^{-1}\sum_{n=1}^{\infty}\overline{\phi_{n+1}\left(\frac{b^2}{z}\right)}+\frac{b^2(b^2-a^2)\Lambda_{cb}^{-1}}{z^3}\sum_{n=1}^{\infty}\overline{\omega_{n+1}'}\left(\frac{b^2}{z}\right)\\ \quad+\frac{\Lambda_{cb}^{-1}(b^2-a^2)}{z}\sum_{n=1}^{\infty}C_{n+1}-\frac{b^2}{z}\sum_{n=1}^{\infty}\overline{C_{n+1}}, \quad z\in S_b\\ \omega_0(z)+\Lambda_{bc}\overline{\phi_0}\left(\frac{b^2}{z}\right)-\overline{C_0}\frac{b^2}{z}+(1+\Pi_{cb})\overline{C_1}\frac{b^2}{z}+(1+\Pi_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\phi_{n+1}}\left(\frac{b^2}{z}\right), \quad z\in S_c\end{cases} \tag{45b}$$

where the recurrence formulae for $\phi_n(z)$ and $\omega_n(z)$ are

$$\phi_{n+1}(z)=\begin{cases}(1+\Lambda_{bc})\phi_0(z)+\Pi_{cb}C_1z, \quad n=0\\ \Pi_{cb}\Lambda_{ab}\phi_n\left(\frac{a^2}{b^2}z\right)+\frac{a^2(a^2-b^2)\Pi_{cb}\Pi_{ab}z^3}{b^6}\left[\omega_n'\left(\frac{a^2}{b^2}z\right)-\frac{b^4}{a^4}\frac{(a^2-b^2)}{z^2}\phi_n'\left(\frac{a^2}{b^2}z\right)+\frac{b^2}{a^2}\frac{(a^2-b^2)}{z}\phi_n''\left(\frac{a^2}{b^2}z\right)+\frac{\overline{C_n}}{a^2}\frac{b^4}{z^2}\right]\\ \quad+\frac{\Pi_{cb}(\Lambda_{ab}+\Pi_{ba})}{1-\Pi_{ba}}\frac{a^2}{b^2}C_nz+\frac{\Pi_{cb}\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi_{ba}}\frac{a^2}{b^2}\overline{C_n}z+\Pi_{cb}C_{n+1}z, \quad n=1,2,3,\ldots\end{cases} \tag{46a}$$

$$\omega_{n+1}(z)=\begin{cases}(1+\Pi_{bc})\omega_0(z)-(1+\Pi_{bc})\overline{C_0}\frac{b^2}{z}+\overline{C_1}\frac{b^2}{z}, \quad n=0\\ \Lambda_{cb}\Pi_{ab}\left[\omega_n\left(\frac{a^2}{b^2}z\right)+\frac{a^2-b^2}{a^2}\frac{b^2}{z}\phi_n'\left(\frac{a^2}{b^2}z\right)-\overline{C_n}\frac{b^2}{z}\right]+\overline{C_{n+1}}\frac{b^2}{z}, \quad n=1,2,3,\ldots\end{cases} \tag{46b}$$

with $C_n=\overline{\phi_n'}(0)$.

For a special case when material a and material b are the same, Eq. (45) reduces to

$$\phi(z)=\begin{cases}(1+\Lambda_{bc})\phi_0(z)+\Pi_{cb}C_1z, \quad z\in S_b\\ \phi_0(z)+\Pi_{bc}\overline{\omega_0}\left(\frac{b^2}{z}\right)-\Pi_{bc}C_0z, \quad z\in S_c\end{cases} \tag{47a}$$

$$\omega(z)=\begin{cases}(1+\Pi_{bc})\omega_0(z)-(1+\Pi_{bc})\overline{C_0}\frac{b^2}{z}+\overline{C_1}\frac{b^2}{z}, \quad z\in S_b\\ \omega_0(z)+\Lambda_{bc}\overline{\phi_0}\left(\frac{b^2}{z}\right)-\overline{C_0}\frac{b^2}{z}+(1+\Pi_{cb})\overline{C_1}\frac{b^2}{z}, \quad z\in S_c\end{cases} \tag{47b}$$

where

$$C_0=\frac{F\mathrm{e}^{-\mathrm{i}\gamma}}{2\pi(1+\kappa_c)}\frac{1}{\overline{z_0}}$$

$$C_1=\frac{(1+\Lambda_{bc})(\Pi_{cb}\overline{C_0}+C_0)}{1-\Pi_{cb}^2}$$

which is the solution to the corresponding single inclusion problem (Honein and Herrmann, 1990).

Putting $\Lambda_{bc}=\Pi_{bc}=-1$ and substituting Eqs. (7) and (15) into (47), the solution to the corresponding hole problem under a point force is obtained as

$$\phi(z)=\begin{cases}0, \quad z\in S_b\\ \frac{-F}{2\pi(1+\kappa_c)}\left[\mathrm{e}^{\mathrm{i}\gamma}\log(z-z_0)+\kappa_c\mathrm{e}^{\mathrm{i}\gamma}\log\left(\frac{b^2}{z}-\overline{z_0}\right)-\frac{\mathrm{e}^{-\mathrm{i}\gamma}(z-z_0)}{\left(\frac{b^2}{z}-\overline{z_0}\right)}-\frac{z\mathrm{e}^{-\mathrm{i}\gamma}}{\overline{z_0}}\right], \quad z\in S_c\end{cases} \tag{48a}$$

$$\omega(z)=\begin{cases}0, \quad z\in S_b\\ \frac{-F}{2\pi(1+\kappa_c)}\left[\frac{b^2}{z}\frac{\mathrm{e}^{\mathrm{i}\gamma}}{(z-z_0)}-\kappa_c\mathrm{e}^{-\mathrm{i}\gamma}\log(z-z_0)-\frac{\mathrm{e}^{\mathrm{i}\gamma}z_0}{(z-\overline{z_0})}-\mathrm{e}^{-\mathrm{i}\gamma}\log\left(\frac{b^2}{z}-\overline{z_0}\right)-\frac{z\mathrm{e}^{-\mathrm{i}\gamma}}{z_0}\right], \quad z\in S_c\end{cases} \tag{48b}$$

which is in agreement with the result obtained by Honein and Herrmann (1988).

3.2. Case II: A point force embedded in S_b

When singularity or a point force is embedded in S_b, the problem becomes more difficult to solve. To satisfy the single-valued conditions of displacements and traction, the stress functions must have the form

$$\phi(z) = \begin{cases} \sum_{n=1}^{\infty} \phi_{an}(z), & z \in S_a \\ \phi_0(z) + \sum_{n=1}^{\infty} [\phi_n(z) + \phi_{bn}(z)], & z \in S_b \\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)} \log \frac{z}{b} + \phi_{c0}(z) + \sum_{n=1}^{\infty} \phi_{cn}(z), & z \in S_c \end{cases} \tag{49a}$$

$$\omega(z) = \begin{cases} \sum_{n=1}^{\infty} \omega_{an}(z), & z \in S_a \\ \omega_0(z) + \sum_{n=1}^{\infty} [\omega_n(z) + \omega_{bn}(z)], & z \in S_b \\ \frac{\kappa_c Fe^{-i\gamma}}{2\pi(1+\kappa_c)} \log \frac{z}{b} + \omega_{c0}(z) + \sum_{n=1}^{\infty} \omega_{cn}(z), & z \in S_c \end{cases} \tag{49b}$$

By the same arguments as in case I, the alternating technique is applied to solve the current problem. First, we regard regions S_a and S_b composed of the same material b and region S_c of material c. $\phi_1(z)$ and $\omega_1(z)$ holomorphic (except at $z = 0$) in $S_a \cup S_b$, $\phi_{c0}(z)$and $\omega_{c0}(z)$ holomorphic in S_c are introduced to satisfy the continuity of traction and displacement across L that

$$\phi_1(\rho) + \overline{\omega_1(\rho)} + \phi_0^*(\rho) + \overline{\omega_0^*(\rho)} = \phi_{c0}(\rho) + \overline{\omega_{c0}(\rho)} \tag{50}$$

$$\frac{1}{G_b}[\kappa_b \phi_1(\rho) - \overline{\omega_1(\rho)} + \kappa_b \phi_0^*(\rho) - \overline{\omega_0^*(\rho)}] = \frac{1}{G_c}[\kappa_c \phi_{c0}(\rho) - \overline{\omega_{c0}(\rho)}] \tag{51}$$

where

$$\phi_0^*(z) = -\frac{Fe^{i\gamma}}{2\pi(1+\kappa_b)} \log\left(b - \frac{bz_0}{z}\right) \tag{52}$$

$$\omega_0^*(z) = \frac{F}{2\pi(1+\kappa_b)} \left[\left(\overline{z_0} - \frac{b^2}{z}\right)\frac{e^{i\gamma}}{z - z_0} + \kappa_b e^{-i\gamma} \log\left(b - \frac{bz_0}{z}\right)\right] \tag{53}$$

By the standard analytical continuation arguments it follows that

$$\overline{\omega_1}\left(\frac{b^2}{z}\right) + \phi_0^*(z) - \phi_{c0}(z) - C_1 z = 0, \quad z \in S_c \tag{54}$$

$$\overline{\omega_{c0}}\left(\frac{b^2}{z}\right) - \phi_1(z) - \overline{\omega_0^*}\left(\frac{b^2}{z}\right) - C_1 z = 0, \quad z \in S_a \cup S_b \tag{55}$$

$$\frac{\kappa_b \phi_0^*(z) - \overline{\omega_1}\left(\frac{b^2}{z}\right)}{G_b} - \frac{\kappa_c \phi_{c0}(z)}{G_c} + \frac{C_1 z}{G_b} = 0, \quad z \in S_c \tag{56}$$

$$\frac{\overline{\omega_0^*}\left(\frac{b^2}{z}\right) - \kappa_b \phi_1(z)}{G_b} - \frac{\overline{\omega_{c0}}\left(\frac{b^2}{z}\right)}{G_c} + \frac{C_1 z}{G_b} = 0, \quad z \in S_a \cup S_b \tag{57}$$

Decoupling of Eqs. (54)–(57) yields

$$\phi_1(z) = \Pi_{cb}\overline{\omega_0^*}\left(\frac{b^2}{z}\right) + \Pi_{cb}C_1 z, \quad z \in S_a \cup S_b \tag{58}$$

$$\omega_1(z) = \Lambda_{cb}\overline{\phi_0^*}\left(\frac{b^2}{z}\right) + \overline{C_1}\frac{b^2}{z}, \quad z \in S_a \cup S_b \tag{59}$$

$$\phi_{c0}(z) = (1 + \Lambda_{cb})\phi_0^*(z), \quad z \in S_c \tag{60}$$

$$\omega_{c0}(z) = (1 + \Pi_{cb})\omega_0^*(z) + (1 + \Pi_{cb})\overline{C_1}\frac{b^2}{z}, \quad z \in S_c \tag{61}$$

where $C_1 = \overline{\phi_1'}(0)$.

Since this result is based on the assumption that region S_a is made up of material b, it cannot satisfy the continuity condition across L^*, which lies between material a and b.

In the second step, we assume region S_b and S_c be made up of the same material b and region S_a of material a. Additional terms $\phi_{b1}(z)$ and $\omega_{b1}(z)$ holomorphic in $S_b \cup S_c$, $\omega_{a1}(z)$ and $\omega_{a1}(z)$ holomorphic (except at $z = 0$) in S_a are introduced to satisfy the continuity conditions across L^* that

$$\phi_{a1}(\sigma) + \overline{\omega_{a1}(\sigma)} = \phi_1(\sigma) + \phi_{b1}(\sigma) + \overline{\omega_1^a(\sigma)} + \overline{\omega_{b1}(\sigma)} + \phi_0(\sigma) + \overline{\omega_0(\sigma)} \tag{62}$$

$$\frac{\kappa_a\phi_{a1}(\sigma) - \overline{\omega_{a1}(\sigma)}}{G_a} = \frac{\kappa_b[\phi_{b1}(\sigma) + \phi_1(\sigma) + \phi_0(\sigma)] - [\overline{\omega_{b1}(\sigma)} + \overline{\omega_1^a(\sigma)} + \overline{\omega_0(\sigma)}]}{G_b} \tag{63}$$

where $\omega_1^a(z) = \omega_1(z) + \frac{(a^2-b^2)}{z}\phi_1'(z)$.

Based on the method of analytical continuation, we have

$$\phi_{a1}(z) - \phi_1(z) - \overline{\omega_{b1}}\left(\frac{a^2}{z}\right) - \phi_0(z) - C_0 z - C_1 z + C_{a1} z = 0, \quad z \in S_a \tag{64}$$

$$\phi_{b1}(z) + \overline{\omega_1^a}\left(\frac{a^2}{z}\right) + \overline{\omega_0}\left(\frac{a^2}{z}\right) - \overline{\omega_{a1}}\left(\frac{a^2}{z}\right) - C_0 z - C_1 z + C_{a1} z = 0, \quad z \in S_b \cup S_c \tag{65}$$

$$\frac{\kappa_a\phi_{a1}(z)}{G_a} - \frac{\kappa_b\phi_1(z) + \kappa_b\phi_0(z) - \overline{\omega_{b1}}\left(\frac{a^2}{z}\right)}{G_b} + \frac{C_0 z}{G_b} + \frac{C_1 z}{G_b} - \frac{C_{a1} z}{G_a} = 0, \quad z \in S_a \tag{66}$$

$$\frac{\kappa_b\phi_{b1}(z) - \overline{\omega_1^a}\left(\frac{a^2}{z}\right) - \overline{\omega_0}\left(\frac{a^2}{z}\right)}{G_b} + \frac{\overline{\omega_{a1}}\left(\frac{a^2}{z}\right)}{G_a} + \frac{C_0 z}{G_b} + \frac{C_1 z}{G_b} - \frac{C_{a1} z}{G_a} = 0, \quad z \in S_b \cup S_c \tag{67}$$

Solve Eqs. (64)–(67) to obtain

$$\phi_{a1}(z) = (1 + \Lambda_{ab})[\phi_1(z) + \phi_0(z)] + \Pi_{ba}(C_{a1} - C_0)z, \quad z \in S_a \tag{68}$$

$$\omega_{a1}(z) = (1 + \Pi_{ab})[\omega_0(z) + \omega_1^a(z)] - (1 + \Pi_{ab})\overline{C_1}\frac{a^2}{z} + (\overline{C_{a1}} - \overline{C_0})\frac{a^2}{z}, \quad z \in S_a \tag{69}$$

$$\phi_{b1}(z) = \Pi_{ab}\left[\overline{\omega_0}\left(\frac{a^2}{z}\right) + \overline{\omega_1^a}\left(\frac{a^2}{z}\right)\right] - \Pi_{ab}(C_0 + C_1)z, \quad z \in S_b \cup S_c \tag{70}$$

$$\omega_{b1}(z) = \Lambda_{ab}\left[\overline{\phi_0}\left(\frac{a^2}{z}\right) + \overline{\phi_1}\left(\frac{a^2}{z}\right)\right] - (\overline{C_0} + \overline{C_1})\frac{a^2}{z} + (1 + \Pi_{ba})\overline{C_{a1}}\frac{a^2}{z}, \quad z \in S_b \cup S_c \tag{71}$$

where $C_0 = \overline{\phi_0'}(0)$ and $C_{a1} = \overline{\phi_{a1}'}(0)$.

Since this result is based on the assumption that region S_c is made up of material b, it cannot satisfy the continuity conditions across L.

In the third step, we again regard regions S_a and S_b composed of the same material b and region S_c of material c. Additional terms $\phi_2(z)$, $\omega_2(z)$ holomorphic (except at $z = 0$) in $S_a \cup S_b$ and $\phi_{c1}(z)$, $\omega_{c1}(z)$ holomorphic in S_c are introduced to satisfy the continuity conditions across L that

$$\phi_{b1}(\rho) + \overline{\omega^b_{b1}(\rho)} + \phi_2(\rho) + \overline{\omega_2(\rho)} = \phi_{c1}(\rho) + \overline{\omega_{c1}(\rho)} \tag{72}$$

$$\frac{1}{G_b}\{\kappa_b[\phi_{b1}(\rho) + \phi_2(\rho)] - [\overline{\omega^b_{b1}(\rho)} + \overline{\omega_2(\rho)}]\} = \frac{1}{G_c}[\kappa_c\phi_{c1}(\rho) - \overline{\omega_{c1}(\rho)}] \tag{73}$$

where $\omega^b_{b1}(z) = \omega_{b1}(z) + \frac{(b^2-a^2)}{z}\phi'_{b1}(z)$.

By applying the standard analytical continuation arguments, it follows that

$$\phi_{b1}(z) + \overline{\omega_2}\left(\frac{b^2}{z}\right) - \phi_{c1}(z) - C_2 z = 0, \quad z \in S_c \tag{74}$$

$$\overline{\omega_{c1}}\left(\frac{b^2}{z}\right) - \phi_2(z) - \overline{\omega^b_{b1}}\left(\frac{b^2}{z}\right) - C_2 z = 0, \quad S_a \cup S_b \tag{75}$$

$$\frac{1}{G_b}\left[\kappa_b\phi_{b1}(z) - \overline{\omega_2}\left(\frac{b^2}{z}\right)\right] - \frac{\kappa_c}{G_c}\phi_{c1}(z) + \frac{C_2 z}{G_b} = 0, \quad z \in S_c \tag{76}$$

$$-\frac{1}{G_c}\overline{\omega_{c1}}\left(\frac{b^2}{z}\right) - \frac{1}{G_b}\left[\kappa_b\phi_2(z) - \overline{\omega^b_{b1}}\left(\frac{b^2}{z}\right)\right] + \frac{C_2 z}{G_b} = 0, \quad S_a \cup S_b \tag{77}$$

Decoupling of Eqs. (74)–(77) yields

$$\phi_2(z) = \Pi_{cb}\overline{\omega^b_{b1}}\left(\frac{b^2}{z}\right) + \Pi_{cb}C_2 z, \quad z \in S_a \cup S_b \tag{78}$$

$$\omega_2(z) = \Lambda_{cb}\overline{\phi_{b1}}\left(\frac{b^2}{z}\right) + \overline{C_2}\frac{b^2}{z}, \quad z \in S_a \cup S_b \tag{79}$$

$$\phi_{c1}(z) = (1 + \Lambda_{cb})\phi_{b1}(z), \quad z \in S_c \tag{80}$$

$$\omega_{c1}(z) = (1 + \Pi_{cb})\omega^b_{b1}(z) + (1 + \Pi_{cb})\overline{C_2}\frac{b^2}{z}, \quad z \in S_c \tag{81}$$

where $C_2 = \overline{\phi'_2}(0)$.

In the fourth step, regions S_b and S_c are assumed to make up with material b again. Repetitions of second and third step, the analytical continuation method is alternatively applied to two interfaces to obtain the additional terms $\phi_{an}(z)$, $\phi_{bn}(z)$, $\phi_{cn}(z)$, $\phi_{n+1}(z)$ and $\omega_{an}(z)$, $\omega_{bn}(z)$, $\omega_{cn}(z)$, $\omega_{n+1}(z)$ ($n = 2, 3, 4, \ldots$). The complete stress functions can be finally obtained as

$$\phi(z) = \begin{cases} (1 + \Lambda_{ab})\phi_0(z) + (1 + \Lambda_{ab})\sum_{n=1}^{\infty}\phi_n(z) - \Pi_{ba}C_0 z + \frac{\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi^2_{ba}}(C_0 + \Pi_{ba}\overline{C_0})z \\ \quad + \frac{\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi^2_{ba}}\sum_{n=1}^{\infty}(C_n + \Pi_{ba}\overline{C_n})z, \quad z \in S_a \\ \phi_0(z) + \Pi_{ab}\overline{\omega_0}\left(\frac{a^2}{z}\right) + \sum_{n=1}^{\infty}\phi_n(z) - \Pi_{ab}C_0 z + \Lambda^{-1}_{cb}\sum_{n=1}^{\infty}\left[\overline{\omega_{n+1}}\left(\frac{b^2}{z}\right) - C_{n+1}z\right], \quad z \in S_b \\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b} + (1 + \Lambda_{cb})\phi^*_0(z) + (1 + \Lambda^{-1}_{cb})\sum_{n=1}^{\infty}\left[\overline{\omega_{n+1}}\left(\frac{b^2}{z}\right) - C_{n+1}z\right], \quad z \in S_c \end{cases} \tag{82a}$$

$$\omega(z)=\begin{cases}(1+\Pi_{ab})\left[\omega_0(z)+\sum_{n=1}^{\infty}\omega_n(z)+\frac{a^2-b^2}{z}\sum_{n=1}^{\infty}\phi_n'(z)\right]+\sum_{n=1}^{\infty}\frac{(1+\Lambda_{ab})(\Pi_{ba}C_n+\overline{C_n})}{1-\Pi_{ba}}\frac{a^2}{z}\\ \quad+\frac{(1+\Lambda_{ab})(\Pi_{ba}C_0+\overline{C_0})}{1-\Pi_{ba}^2}\frac{a^2}{z}-\overline{C_0}\frac{a^2}{z}-(1+\Pi_{ba})\sum_{n=1}^{\infty}\overline{C_n}\frac{a^2}{z}, \quad z\in S_a\\ \omega_0(z)+\sum_{n=1}^{\infty}\omega_n(z)+\Lambda_{ab}\overline{\phi_0}\left(\frac{a^2}{z}\right)+\frac{(1+\Lambda_{ab})(\Pi_{ba}C_0+\overline{C_0})}{1-\Pi_{ba}^2}\frac{a^2}{z}+\Pi_{cb}^{-1}\sum_{n=1}^{\infty}\overline{\phi_{n+1}}\left(\frac{b^2}{z}\right)\\ \quad+\Lambda_{cb}^{-1}\frac{b^2(b^2-a^2)}{z^3}\sum_{n=1}^{\infty}\overline{\omega_{n+1}'}\left(\frac{b^2}{z}\right)-\overline{C_0}\frac{a^2}{z}-\sum_{n=1}^{\infty}\overline{C_{n+1}}\frac{b^2}{z}+\frac{b^2-a^2}{z}\Lambda_{cb}^{-1}\sum_{n=1}^{\infty}C_{n+1}, \quad z\in S_b\\ \frac{\kappa_c F e^{-i\gamma}}{2\pi(1+\kappa_c)}\log z+(1+\Pi_{cb})\omega_0^*(z)+(1+\Pi_{cb})\overline{C_1}\frac{b^2}{z}+(1+\Pi_{cb}^{-1})\sum_{n=1}^{\infty}\overline{\phi_{n+1}}\left(\frac{b^2}{z}\right), \quad z\in S_c\end{cases} \tag{82b}$$

where the recurrence formulae for $\phi_n(z)$ and $\omega_n(z)$ are

$$\phi_{n+1}(z)=\begin{cases}\Pi_{cb}\overline{\omega_0^*}\left(\frac{b^2}{z}\right)+\Pi_{cb}C_1 z, \quad n=0\\ \Pi_{cb}\Lambda_{ab}\phi_n\left(\frac{a^2}{b^2}z\right)+\frac{a^2(a^2-b^2)\Pi_{cb}\Pi_{ab}z^3}{b^6}\left[\omega_n'\left(\frac{a^2}{b^2}z\right)-\frac{b^4}{a^4}\frac{(a^2-b^2)}{z^2}\phi_n'\left(\frac{a^2}{b^2}z\right)\right.\\ \quad\left.+\frac{b^2}{a^2}\frac{(a^2-b^2)}{z}\phi_n''\left(\frac{a^2}{b^2}z\right)+\frac{\overline{C_n}}{a^2}\frac{b^4}{z^2}\right]+\frac{\Pi_{cb}(\Lambda_{ab}+\Pi_{ba})}{1-\Pi_{ba}}\frac{a^2}{b^2}C_n z\\ \quad+\frac{\Pi_{cb}\Pi_{ba}(1+\Lambda_{ab})}{1-\Pi_{ba}}\frac{a^2}{b^2}\overline{C_n}z+\Pi_{cb}C_{n+1}z, \quad n=1,2,3,\ldots\end{cases} \tag{83a}$$

$$\omega_{n+1}(z)=\begin{cases}\Lambda_{cb}\overline{\phi_0^*}\left(\frac{b^2}{z}\right)+\overline{C_1}\frac{b^2}{z}, \quad n=0\\ \Lambda_{cb}\Pi_{ab}\left[\omega_n\left(\frac{a^2}{b^2}z\right)+\frac{a^2-b^2}{a^2}\frac{b^2}{z}\phi_n'\left(\frac{a^2}{b^2}z\right)-\overline{C_n}\frac{b^2}{z}\right]+\overline{C_{n+1}}\frac{b^2}{z}, \quad n=1,2,3,\ldots\end{cases} \tag{83b}$$

with $C_n=\overline{\phi_n'}(0)$.

For the special case that material a and material b are the same, Eq. (82) reduces to

$$\phi(z)=\begin{cases}-\frac{Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log(z-z_0)+\frac{\Pi_{cb}F}{2\pi(1+\kappa_b)}\left[\frac{(z_0-z)e^{-i\gamma}}{b^2/z-\overline{z_0}}+\kappa_b e^{i\gamma}\log\left(b-\frac{\overline{z_0}z}{b}\right)\right]\\ \quad+\frac{\Pi_{cb}^2F\left[(\Pi_{cb}-\kappa_b)z_0e^{-i\gamma}+(1-\Pi_{cb}\kappa_b)\overline{z_0}e^{i\gamma}\right]}{2\pi(1+\kappa_b)(1-\Pi_{cb}^2)}\frac{z}{b^2}, \quad z\in S_b\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}-\frac{(1+\Lambda_{cb})Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log\left(b-\frac{bz_0}{z}\right), \quad z\in S_c\end{cases} \tag{84a}$$

$$\omega(z)=\begin{cases}\frac{F}{2\pi(1+\kappa_b)}\left[\left(\overline{z_0}-\frac{b^2}{z}\right)\frac{e^{i\gamma}}{(z-z_0)}+\kappa_b e^{-i\gamma}\log(z-z_0)-\Lambda_{cb}e^{-i\gamma}\log\left(b-\frac{\overline{z_0}z}{b}\right)\right]\\ \quad+\frac{\Pi_{cb}F\left[(1-\Pi_{cb}\kappa_b)e^{-i\gamma}z_0+(\Pi_{cb}-\kappa_b)e^{i\gamma}\overline{z_0}\right]}{2\pi(1+\kappa_b)(1-\Pi_{cb}^2)}\frac{1}{z}, \quad z\in S_b\\ \frac{\kappa_c Fe^{-i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+\frac{(1+\Pi_{cb})F}{2\pi(1+\kappa_b)}\left[\left(\overline{z_0}-\frac{b^2}{z}\right)\frac{e^{i\gamma}}{(z-z_0)}+\kappa_b e^{-i\gamma}\log\left(b-\frac{bz_0}{z}\right)\right]\\ \quad+\frac{\Pi_{cb}(1+\Pi_{cb})F\left[(1-\Pi_{cb}\kappa_b)e^{-i\gamma}z_0+(\Pi_{cb}-\kappa_b)e^{i\gamma}\overline{z_0}\right]}{2\pi(1+\kappa_b)(1-\Pi_{cb}^2)}\frac{1}{z}, \quad z\in S_c\end{cases} \tag{84b}$$

which is the solution to the corresponding single inclusion problem (Honein and Herrmann, 1990).

If a point force is assumed to situate at origin, Eq. (84) can be further simplified to

$$\phi(z)=\begin{cases}-\frac{Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log z-\frac{\Pi_{cb}Fe^{-i\gamma}}{2\pi(1+\kappa_b)}\frac{z^2}{b^2}+\frac{\Pi_{cb}\kappa_bFe^{i\gamma}}{2\pi(1+\kappa_b)}\log b, \quad z\in S_b\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}-\frac{(1+\Lambda_{cb})Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log b, \quad z\in S_c\end{cases} \tag{85a}$$

$$\omega(z)=\begin{cases}-\frac{Fe^{i\gamma}}{2\pi(1+\kappa_b)}\frac{b^2}{z^2}+\frac{\kappa_b Fe^{-i\gamma}}{2\pi(1+\kappa_b)}\log z-\frac{\Lambda_{cb}e^{-i\gamma}F}{2\pi(1+\kappa_b)}\log b, & z\in S_b\\ \frac{\kappa_c Fe^{-i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+\frac{(1+\Pi_{cb})F}{2\pi(1+\kappa_b)}\left[-\frac{b^2e^{i\gamma}}{z^2}+\kappa_b e^{-i\gamma}\log b\right], & z\in S_c\end{cases}\tag{85b}$$

which is in agreement with the results provided by Dundurs (1963).

3.3. Case III: A point force embedded in S_a

When singularity or a point force is embedded in S_a, similar to the case II with singularity embedded in S_b, the stress functions must have the form

$$\phi(z)=\begin{cases}\phi_0(z)+\phi_{a0}(z)+\sum_{n=1}^{\infty}\phi_{an}(z), & z\in S_a\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a}+\sum_{n=1}^{\infty}[\phi_n(z)+\phi_{bn}(z)], & z\in S_b\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+\sum_{n=1}^{\infty}\phi_{cn}(z), & z\in S_c\end{cases}\tag{86a}$$

$$\omega(z)=\begin{cases}\omega_0(z)+\omega_{a0}(z)+\sum_{n=1}^{\infty}\omega_{an}(z), & z\in S_a\\ \frac{\kappa_b Fe^{-i\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a}+\sum_{n=1}^{\infty}[\omega_n(z)+\omega_{bn}(z)], & z\in S_b\\ \frac{\kappa_c Fe^{-i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+\frac{\kappa_b G_c Fe^{-i\gamma}}{\pi G_b(1+\kappa_b)}\log\frac{b}{a}+\sum_{n=1}^{\infty}\omega_{cn}(z), & z\in S_c\end{cases}\tag{86b}$$

By a way similar to the previous approach, we can find

$$\phi(z)=\begin{cases}\phi_0(z)+\phi_{a0}(z)+(1+\Lambda_{ab}^{-1})\sum_{n=1}^{\infty}\overline{\omega_{n+1}}\left(\frac{a^2}{z}\right)-(1+\Lambda_{ab}^{-1})\sum_{n=1}^{\infty}C_n z\\ \quad-\frac{(1+\Lambda_{ab}^{-1}+\Lambda_{ab}^{-1}\Pi_{ba})(1+\Lambda_{ab})}{1-\Pi_{ba}^2}\left(\Pi_{ba}\sum_{n=1}^{\infty}\overline{C_n}+\sum_{n=1}^{\infty}C_n\right)z, & z\in S_a\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a}+\sum_{n=1}^{\infty}\phi_n(z)+\Lambda_{ab}^{-1}\sum_{n=1}^{\infty}\overline{\omega_{n+1}}\left(\frac{a^2}{z}\right)-\frac{(1+\Lambda_{ab}^{-1})}{1-\Pi_{ba}}\sum_{n=1}^{\infty}[(\Pi_{ba}\overline{C_n}+C_n)z]-\Lambda_{ab}^{-1}\sum_{n=1}^{\infty}C_n z, & z\in S_b\\ \frac{-Fe^{i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+(1+\Lambda_{cb})\sum_{n=1}^{\infty}\phi_n(z), & z\in S_c\end{cases}\tag{87a}$$

$$\omega(z)=\begin{cases}\omega_0(z)+\omega_{a0}(z)+(1+\Pi_{ab}^{-1})\sum_{n=1}^{\infty}\overline{\phi_{n+1}}\left(\frac{a^2}{z}\right)+\frac{(1+\Lambda_{ab})}{1-\Pi_{ba}^2}\sum_{n=1}^{\infty}(\overline{C_n}+\Pi_{ba}C_n)\frac{a^2}{z}, & z\in S_a\\ \frac{\kappa_b Fe^{-i\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a}+\sum_{n=1}^{\infty}\omega_n(z)+\Pi_{ab}^{-1}\sum_{n=1}^{\infty}\overline{\phi_{n+1}}\left(\frac{a^2}{z}\right)+\frac{a^2(a^2-b^2)}{z^3}\Lambda_{ab}^{-1}\sum_{n=1}^{\infty}\overline{\omega'_{n+1}}\left(\frac{a^2}{z}\right)\\ \quad+\sum_{n=1}^{\infty}\overline{C_n}\frac{a^2}{z}+\frac{a^2-b^2}{z}\Lambda_{ab}^{-1}\left[\frac{(1+\Lambda_{ab})\Pi_{ba}}{1-\Pi_{ba}}\sum_{n=1}^{\infty}\overline{C_n}+\frac{(2+\Lambda_{ab}-\Pi_{ba})}{1-\Pi_{ba}}\sum_{n=1}^{\infty}C_n\right], & z\in S_b\\ \frac{\kappa_c Fe^{-i\gamma}}{2\pi(1+\kappa_c)}\log\frac{z}{b}+\frac{\kappa_b G_c Fe^{-i\gamma}}{\pi G_b(1+\kappa_b)}\log\frac{b}{a}+(1+\Pi_{cb})\sum_{n=1}^{\infty}\omega_n(z)+(1+\Pi_{cb})\frac{(b^2-a^2)}{z}\sum_{n=1}^{\infty}\phi'_n(z)\\ \quad+(1+\Pi_{cb})\sum_{n=1}^{\infty}\overline{C_n}\frac{b^2}{z}, & z\in S_c\end{cases}\tag{87b}$$

The recurrence formulae for $\phi_n(z)$ and $\omega_n(z)$ are

$$\phi_{n+1}(z) = \begin{cases} (1+\Lambda_{ba})\phi_0^*(z), \quad n=0 \\ \Pi_{ab}\Lambda_{cb}\phi_n\left(\frac{b^2}{a^2}z\right) + \frac{\Pi_{cb}\Pi_{ab}b^2(b^2-a^2)z^3}{a^6}\left[\omega_n'\left(\frac{b^2}{a^2}z\right) - \frac{a^4}{b^4}\frac{(b^2-a^2)}{z^2}\phi_n'\left(\frac{b^2}{a^2}z\right) \right. \\ \left. \quad + \frac{a^2}{b^2}\frac{b^2-a^2}{z}\phi_n''\left(\frac{b^2}{a^2}z\right)\right] + \Pi_{ab}\left(\frac{b^2}{a^2}-1\right)(C_n - \Pi_{cb}\overline{C_n})z, \quad n=1,2,3,\ldots \end{cases} \tag{88a}$$

$$\omega_{n+1}(z) = \begin{cases} (1+\Pi_{ba})\omega_0^*(z) + (1+\Pi_{ba})\overline{C_{a0}}\frac{a^2}{z}, \quad n=0 \\ \Lambda_{ab}\left[\Pi_{cb}\omega_n\left(\frac{b^2}{a^2}z\right) + \Pi_{cb}\frac{b^2-a^2}{b^2}\frac{a^2}{z}\phi_n'\left(\frac{b^2}{a^2}z\right) + \Pi_{cb}\overline{C_n}\frac{a^2}{z}\right] \\ \quad + \frac{(1+\Lambda_{ab})(\Pi_{ba}\overline{C_n}+C_n)}{1-\Pi_{ba}}\frac{a^2}{z} + \overline{C_n}\frac{a^2}{z}, \quad n=1,2,3,\ldots \end{cases} \tag{88b}$$

with

$$C_{n+1} = \begin{cases} \frac{\Pi_{cb}(1+\Pi_{ba})F}{2\pi(1+\kappa_a)b^2(1-\Pi_{cb}^2)}[\Pi_{cb}(\mathrm{e}^{-\mathrm{i}\gamma}z_0 - \kappa_a\mathrm{e}^{\mathrm{i}\gamma}\overline{z_0}) + (\mathrm{e}^{\mathrm{i}\gamma}\overline{z_0} - \kappa_a\mathrm{e}^{-\mathrm{i}\gamma}z_0)] \\ \quad + \frac{\Pi_{ba}\Pi_{cb}F}{2\pi(1+\kappa_a)b^2(\Pi_{ba}+1)(1-\Pi_{cb}^2)}[\Pi_{ba}(\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma} - \kappa_a z_0\mathrm{e}^{-\mathrm{i}\gamma}) + (z_0\mathrm{e}^{-\mathrm{i}\gamma} - \kappa_a\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma})] \\ \quad + \frac{\Pi_{ba}\Pi_{cb}^2F}{2\pi(1+\kappa_a)b^2(\Pi_{ba}+1)(1-\Pi_{cb}^2)}[\Pi_{ba}(z_0\mathrm{e}^{-\mathrm{i}\gamma} - \kappa_a\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma}) + (\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma} - \kappa_a z_0\mathrm{e}^{-\mathrm{i}\gamma})], \quad \text{for } n=0 \\ \frac{\Pi_{cb}^2}{(1-\Pi_{cb}^2)}\frac{a^2}{b^2}\left[\frac{(\Lambda_{ab}+\Pi_{ba})\overline{C_n}+(2+\Lambda_{ab}-\Pi_{ba})C_n}{1-\Pi_{ba}}\right] + \frac{\Pi_{cb}}{(1-\Pi_{cb}^2)}\frac{a^2}{b^2}\left[\frac{(\Lambda_{ab}+\Pi_{ba})C_n+(2+\Lambda_{ab}-\Pi_{ba})\overline{C_n}}{1-\Pi_{ba}}\right], \quad \text{for } n=1,2,3\ldots \end{cases}$$

where

$$\phi_{a0}(z) = \Pi_{ba}\overline{\omega_0^*}\left(\frac{a^2}{z}\right) + \Pi_{ba}C_{a0}z$$

$$\omega_{a0}(z) = \Lambda_{ba}\overline{\phi_0^*}\left(\frac{a^2}{z}\right) + \overline{C_{a0}}\frac{a^2}{z}$$

$$\omega_0^*(z) = \frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\left(\overline{z_0} - \frac{a^2}{z}\right)\frac{1}{(z-z_0)} + \frac{\kappa_a F\mathrm{e}^{-\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log\left(a - \frac{az_0}{z}\right)$$

$$\phi_0^*(z) = -\frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log\left(a - \frac{az_0}{z}\right)$$

$$C_{a0} = \frac{\Pi_{ba}F}{2\pi(1+\kappa_a)a^2(1-\Pi_{ba}^2)}[(\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma} - \kappa_a z_0\mathrm{e}^{-\mathrm{i}\gamma}) + \Pi_{ba}(z_0\mathrm{e}^{-\mathrm{i}\gamma} - \kappa_a\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma})]$$

$$\phi_0(z) = -\frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log(z-z_0)$$

$$\omega_0(z) = \frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\left(\overline{z_0} - \frac{a^2}{z}\right)\frac{1}{(z-z_0)} + \frac{\kappa_a F\mathrm{e}^{-\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log(z-z_0)$$

For a special case when material b and c are the same, Eq. (87) reduces to

$$\phi(z) = \begin{cases} -\frac{F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log(z-z_0) + \frac{\Pi_{ba}F}{2\pi(1+\kappa_a)}\left[\frac{(z_0-z)\mathrm{e}^{-\mathrm{i}\gamma}}{a^2/z-\overline{z_0}} + \kappa_a\mathrm{e}^{\mathrm{i}\gamma}\log\left(a - \frac{\overline{z_0}z}{a}\right)\right] \\ \quad + \frac{\Pi_{ba}^2F\left[(\Pi_{ba}-\kappa_a)z_0\mathrm{e}^{-\mathrm{i}\gamma}+(1-\Pi_{ba}\kappa_a)\overline{z_0}\mathrm{e}^{\mathrm{i}\gamma}\right]}{2\pi(1+\kappa_a)(1-\Pi_{ba}^2)}\frac{z}{a^2}, \quad z\in S_b \\ \frac{-F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a} - \frac{(1+\Lambda_{ba})F\mathrm{e}^{\mathrm{i}\gamma}}{2\pi(1+\kappa_a)}\log\left(a - \frac{az_0}{z}\right), \quad z\in S_c \end{cases} \tag{89a}$$

$$\omega(z)=\begin{cases}\frac{F}{2\pi(1+\kappa_a)}\left[\left(\overline{z_0}-\frac{a^2}{z}\right)\frac{e^{i\gamma}}{(z-z_0)}+\kappa_a e^{-i\gamma}\log(z-z_0)-\Lambda_{ba}e^{-i\gamma}\log\left(a-\frac{\overline{z_0}z}{a}\right)\right]\\ \quad+\frac{\Pi_{ba}F[(1-\Pi_{ba}\kappa_a)e^{-i\gamma}z_0+(\Pi_{ba}-\kappa_a)e^{i\gamma}\overline{z_0}]}{2\pi(1+\kappa_a)(1-\Pi_{ba}^2)}\frac{1}{z},\quad z\in S_b\\ \frac{\kappa_b F e^{-i\gamma}}{2\pi(1+\kappa_b)}\log\frac{z}{a}+\frac{(1+\Pi_{ba})F}{2\pi(1+\kappa_a)}\left[\left(\overline{z_0}-\frac{a^2}{z}\right)\frac{e^{i\gamma}}{(z-z_0)}+\kappa_a e^{-i\gamma}\log\left(a-\frac{az_0}{z}\right)\right]\\ \quad+\frac{\Pi_{ba}(1+\Pi_{ba})F[(1-\Pi_{ba}\kappa_a)e^{-i\gamma}z_0+(\Pi_{ba}-\kappa_a)e^{i\gamma}\overline{z_0}]}{2\pi(1+\kappa_a)(1-\Pi_{ba}^2)}\frac{1}{z},\quad z\in S_c\end{cases}\tag{89b}$$

which is exactly the same as the expression in Eq. (84) if one replaces Π_{ba}, Λ_{ba}, κ_a, κ_b, a in Eq. (89) with Π_{cb}, Λ_{cb}, κ_b, κ_c, b, respectively.

4. Results and discussion

The stress functions as indicated in Eqs. (45) and (82) are expressed in terms of $\phi_n(z)$ and $\omega_n(z)$ $(n=0,1,2\ldots)$, which may be calculated from a homogeneous solution $\phi_0(z)$ and $\omega_0(z)$ by the recurrence formulae Eqs. (46) and (83). The rate of convergence depends on the ratios $|\phi_{n+1}(z)|/|\phi_n(z)|$ and $|\omega_{n+1}(z)|/|\omega_n(z)|$, which in turn depend on the non-dimensional bimaterial constants Λ_{ab} and Π_{ab} (or Λ_{cb} and Π_{cb}). For most combinations of materials, Λ and Π are less than 1 and 0.5, respectively, which guarantee rapid convergence. Consequently, the convergence rate becomes more rapid as the differences of the elastic constants of the neighboring materials get smaller.

Figs. 2 and 3 respectively show the interfacial normal and shear stress between material b and c for a point force located at S_c. It is seen that the maximum interfacial normal stress increases as a point force is applied closer to the interface as shown in Fig. 2. The interfacial shear stress vanishes at $\theta=90°$ and $\theta=270°$ due to loading symmetry as shown in Fig. 3. Similar trend can also be found for the case of a point force embedded in S_b as shown in Figs. 4 and 5 except that the maximum interfacial normal stress becomes compressive for the present case. When material a is non-existent, the present trimaterial problem is

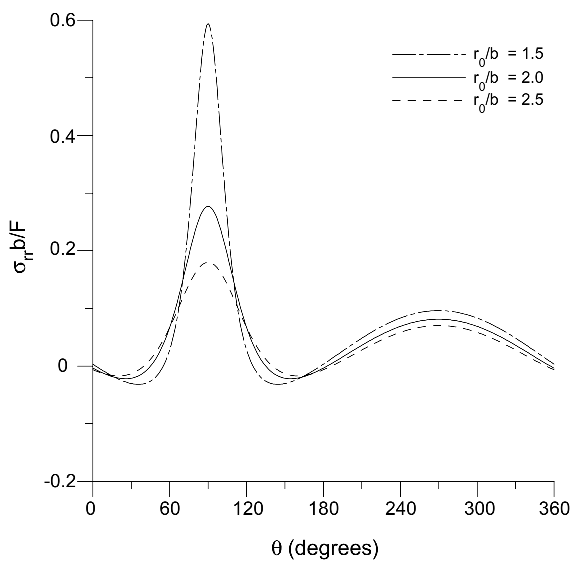

Fig. 2. Angular variations of interfacial normal stress between material b and c for a point force located in S_c ($G_a/G_b=G_c/G_b=1/2$, $\nu_a=\nu_b=\nu_c=0.3$, $b/a=2$, $\theta_0=90°$, $\gamma=90°$).

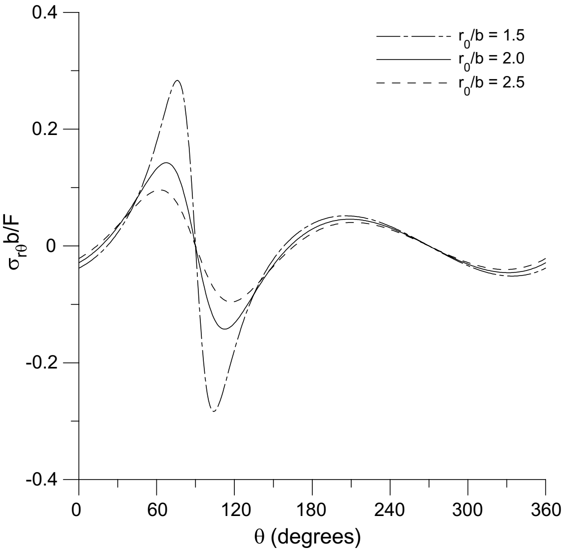

Fig. 3. Angular variations of interfacial shear stress between material b and c for a point force located in S_c ($G_a/G_b = G_c/G_b = 1/2$, $\nu_a = \nu_b = \nu_c = 0.3$, $b/a = 2$, $\theta_0 = 90°$, $\gamma = 90°$).

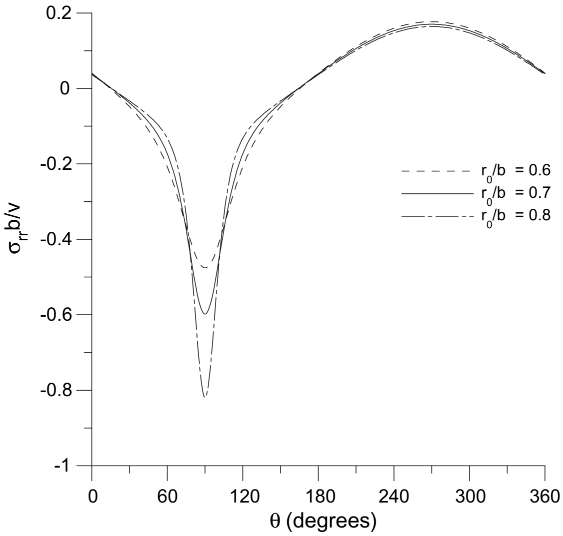

Fig. 4. Angular variations of interfacial normal stress between material b and c for a point force located in S_b ($G_a/G_b = G_c/G_b = 1/2$, $\nu_a = \nu_b = \nu_c = 0.3$, $b/a = 2$, $\theta_0 = 90°, \gamma = 90°$).

degenerated to the thin-layer hole problem whose solution can be obtained by putting $\Lambda_{ab} = \Pi_{ab} = -1$ in Eq. (45) or Eq. (82). The distribution of the interfacial stresses for a thin-layer hole structure with a point force embedded in S_c is shown in Figs. 6 and 7. It is seen that the trend for the present case is nearly the same as that of the trimaterial one, but the magnitude of interfacial stresses for a thin-layer hole structure is less than that of a trimaterial. This is simply because that the interfacial stresses can be further intensified

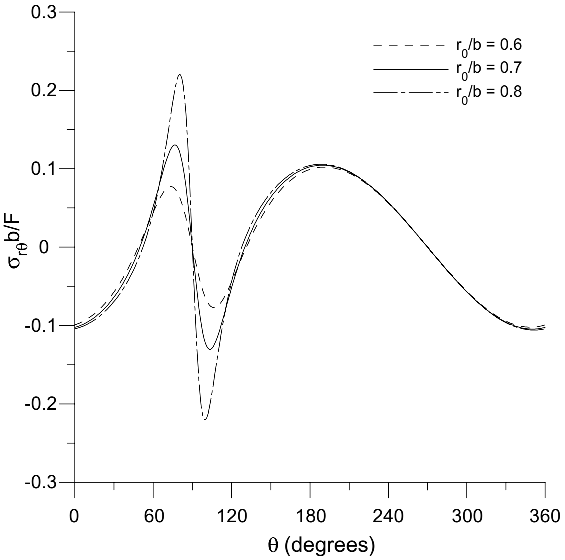

Fig. 5. Angular variations of interfacial shear stress between material b and c for a point force located in S_b ($G_a/G_b = G_c/G_b = 1/2$, $\nu_a = \nu_b = \nu_c = 0.3$, $b/a = 2$, $\theta_0 = 90°, \gamma = 90°$).

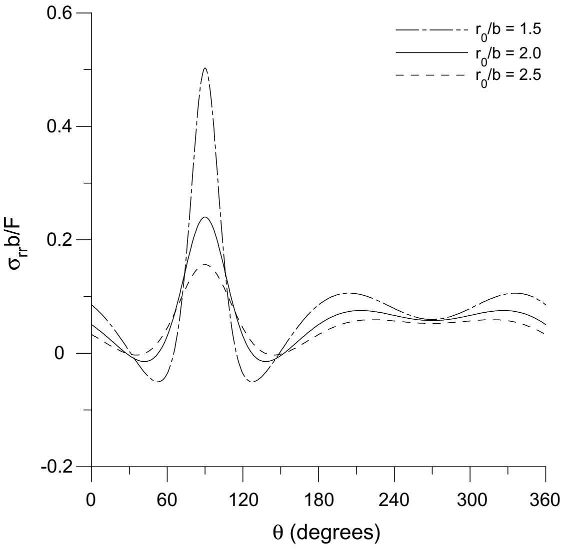

Fig. 6. Angular variations of interfacial normal stress for a thin-layer hole structure for a point force located in S_c ($G_a/G_b = G_c/G_b = 1/2$, $\nu_a = \nu_b = \nu_c = 0.3$, $b/a = 2$, $\theta_0 = 90°, \gamma = 90°$).

(or diminished) by the adjacent material having a higher (or lower) stiffness. Note that all the calculated results shown in Figs. 2–7 are determined by the sum up to $n = 4$ of Eq. Eqs. (45) and (82), since they are checked to achieve a very good convergence of the series form solutions. It is found that the contributions

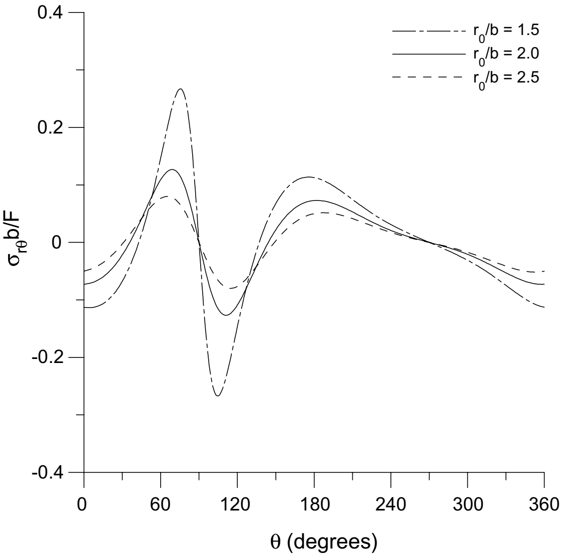

Fig. 7. Angular variations of interfacial shear stress for a thin-layer hole structure for a point force located in S_c ($G_a/G_b = G_c/G_b = 1/2$, $\nu_a = \nu_b = \nu_c = 0.3$, $b/a = 2$, $\theta_0 = 90°, \gamma = 90°$).

of terms with $n = 1, 2, 3$ and 4 to the normal and shear stresses $\sigma_{rr}b/F$ and $\sigma_{r\theta}b/F$ for the case $r_0/b = 2$ in Figs. 2 and 3 are approximately 36.25%, 8.06%, 2.01% and 0.48% respectively. It is likely that the error of approximations with terms up to $n = 4$ is less than 0.5%.

5. Conclusion

An alternative efficient procedure is established to analyze plane elasticity problems of a three-phase circularly cylindrical layered media subject to an arbitrary point force. Within the framework of the procedure of analytical continuation and the method of successive approximations, the solution associated with the heterogeneous problem is sought as transformation on the solution to the corresponding homogeneous problem. It should be emphasized that the method of the present approach can be also extended to solve the problem consisting of any number of layered medium. The convergence rate of the series solution depends on the material combinations in such a way that the convergence rate becomes more rapid if the differences of elastic constants of adjacent materials get smaller.

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